Performance of Random Space-Time Precoded Integer Forcing over Compound MIMO Channels

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Introduction

- The Single-User Multiple-Input Multiple-Output (MIMO) Gaussian channel has been the focus of extensive research

$$\boldsymbol{y}_c = \mathbf{H}_c \boldsymbol{x}_c + \boldsymbol{z}_c,$$

- $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector
- $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector
- $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix
  $\rightarrow$ Fixed over entire block length
- $\boldsymbol{z}_c \sim \mathcal{CSCN}(0, \mathbf{I})$
- Power constraint: $\mathbb{E}(\boldsymbol{x_c}^H \boldsymbol{x_c}) \leq N_t \cdot \mathrm{SNR}$

Introduction

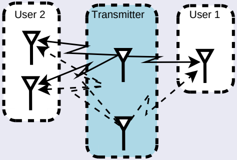
- The MIMO Gaussian broadcast channel has also been widely studied for well over a decade now:

$$\boldsymbol{y}_c^i = \mathbf{H}_c^i \boldsymbol{x}_c + \boldsymbol{z}_c^i$$

- Private (only) Messages vs. Common (only) Messages
  - Capacity is known for both scenarios ✓
  - Practical schemes?
    - Private Message ✓ (DPC: Tomlinson...)
    - Common Message?
      $\Longrightarrow$ Single user: SVD or QR+SIC
      $\Longrightarrow$ Two users: Solved using joint triangularization (Khina '12)
      $\Longrightarrow$ Moderate # of users: Extensions exist, not optimal (Khina '12)
      $\Longrightarrow$ Infinite # of users (knowing only WI-MI): Approximate joint triangularization is not very good $\Longrightarrow$ **Topic of this talk**

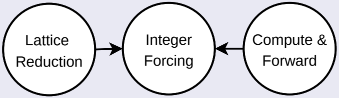
Objective

- Can we find a scheme that is:
  - Practical
    - Linear complexity in the block length
    - Uses off-the-shelf SISO codes
  - Has good provable performance guarantees
  - Universal: Is good for all channels with same WI-MI (compound channel setting), i.e., $\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})$
- Universal $\Longrightarrow$ needs to deal with DoF mismatch

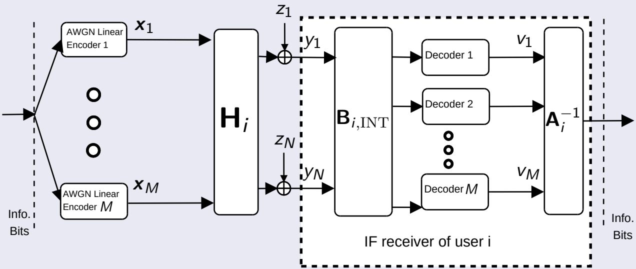


Candidate Scheme for OL-MIMO Broadcast: Integer Forcing

- Equalization scheme introduced by Zhan '10, et. al.



- Idea: Decode linear combination of messages $\Longrightarrow$ Invert



Integer-Forcing Equalization: Basic Idea

- Consider the (SU) channel

$$\mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- At high SNR linear receiver front-end *inverts* the channel (ZF) thus resulting in *noise amplification*

$$\mathbf{H}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Longrightarrow \sigma_1^2 = 2,\ \sigma_2^2 = 5$$

- Can we avoid noise amplification?
- IF idea: If all streams are coded with **same** linear code $\Longrightarrow$

$$\text{Integer} \times \text{Codeword} + \text{Integer} \times \text{Codeword} = \text{Codeword}$$

- However, normal channels do not consist only of integers
- Integer Forcing (IF) equalization equalize the channel to he "nearest" integers-only matrix

Candidate Scheme for OL-MIMO Broadcast: Integer Forcing

- What is already known?
- Ordentlich '15, et. al. (single-user Open-Loop):
  - Rx side - Integer forcing equalization
  - Tx side - **Specific space-time** linear precoding
  - ✓ A linear Non-Vanishing Determinant (NVD) precoder achieves the mutual information up to a constant gap for any channel
  - ☹ Guaranteed gap to capacity is quite large $\Longrightarrow$ doesn't provide satisfactory performance guarantees at moderate rates
- Domanovitz '16, et. al. (single-user Open-Loop):
  - Rx side - Integer forcing equalization
  - Tx side - **Random unitary space-only** linear precoding
  - ✓ Universal bound for scheme outage
- **Random unitary space-time** linear precoding ?

Bad Channels for IF/Linear Equalization

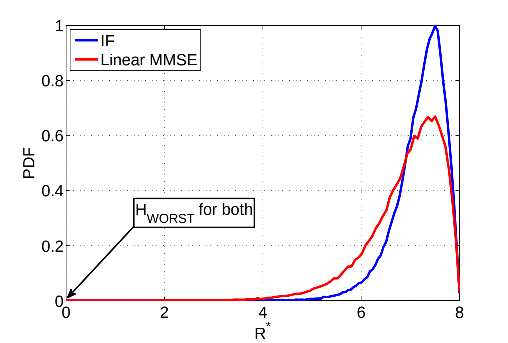


Figure: PDF of 2 × 2 Rayleigh channels normalized to WI=8 bits

- Worst channel $\mathbf{H}_{\text{worst}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$: one stream $\longrightarrow$ 

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

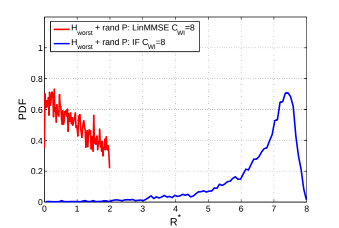


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

- No precoding can salvage linear eq. when channel is singular
- IF copes well with channel being singular

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

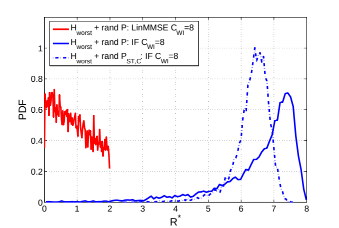


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

- Precoding over a time-extended channel $\Longrightarrow$ "tail" of the PDF decays faster $\Longrightarrow$ improve the WC outage probability.

Compound MIMO Channel Model

- $\mathbf{H}_c$ is part of the compound channel $\mathbb{H}(C_{\mathrm{WI}})$
- Mutual information of the compound channel is maximized by a Gaussian input with covariance matrix $\mathbf{Q}$:
  $C = \max_{\mathbf{Q}:\mathrm{Tr}\,\mathbf{Q}\leq N_t\mathrm{SNR}} \log\det\left(\mathbf{I}_{N_r\times N_r} + \mathbf{H}_c\mathbf{Q}\mathbf{H}_c^T\right)$
- We set $\mathrm{SNR} = 1 \implies \mathbf{H}_c = \mathbf{H}_c\sqrt{\mathrm{SNR}}$, taking $Q = I_{N_t\times N_t} \implies$ $C_{\mathrm{WI}} = \log\det\left(\mathbf{I}_{N_r\times N_r} + \mathbf{H}_c\mathbf{H}_c^T\right)$
- Define:

$$\mathbb{H}(C_{\mathrm{WI}}) = \left\{\mathbf{H}_c \in \mathbb{C}^{N_r\times N_t} : \log\det\left(I + \mathbf{H}_c^T\mathbf{H}_c\right) = C_{\mathrm{WI}}\right\}$$

Compound MIMO Channel Model

- PDF figures $\Longrightarrow$ for **most** precoding matrices good performance, however there is a tail (outage)...
- In contrast to Rayleigh channel all channels in the compound class has same mutual information
  $\Longrightarrow$ Define (**scheme outage**) probability which is taken w.r.t. random precoding ensemble, not w.r.t. to channel statistics
- Instead of constant gap, our target is to bound the worst-case scheme outage. For example, in case of space-only random precoding

$$P_{\text{out}}^{\text{WC}}\left(C_{\text{WI}}, R\right) = \sup_{\mathbf{H}_c \in \mathbb{H}(C_{\text{WI}})} P\left(R_{\text{IF}}\left(\mathbf{H}_c \cdot \mathbf{P}_c\right) < R\right)$$

- When $\mathbf{P}_c$ is drawn from CUE $\Longrightarrow$ channels with equal eigenvalues have equal outage probability

Space-Time Precoding

- A block of T channel uses is processed jointly so that the $N_r \times N_t$ physical MIMO channel is transformed into an aggregate $N_r T \times N_t T$ MIMO channel
- The equivalent channel is

$$\bar{\boldsymbol{y}}_c = \mathcal{H}_c \bar{\boldsymbol{x}}_c + \bar{\boldsymbol{z}}_c$$

where $\bar{\boldsymbol{x}}_c \in \mathbb{C}^{N_t T}$, $\bar{\boldsymbol{y}}_c, \bar{\boldsymbol{z}}_c \in \mathbb{C}^{N_r T}$ and

$$\mathcal{H}_c = \mathbf{I}_{T \times T} \otimes \mathbf{H}_c = \begin{bmatrix} \mathbf{H}_c & 0 & \cdots & 0 \\ 0 & \mathbf{H}_c & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{H}_c \end{bmatrix}$$

Space-Time Precoding

- In our framework, two levels of precoding are applied.
  - $\mathbf{P}_c$ is applied to the physical channel (similar to space-only precoding)
  - $\mathbf{P}_{st,c}$ is applied to the time-extended channel
- The equivalent channel is

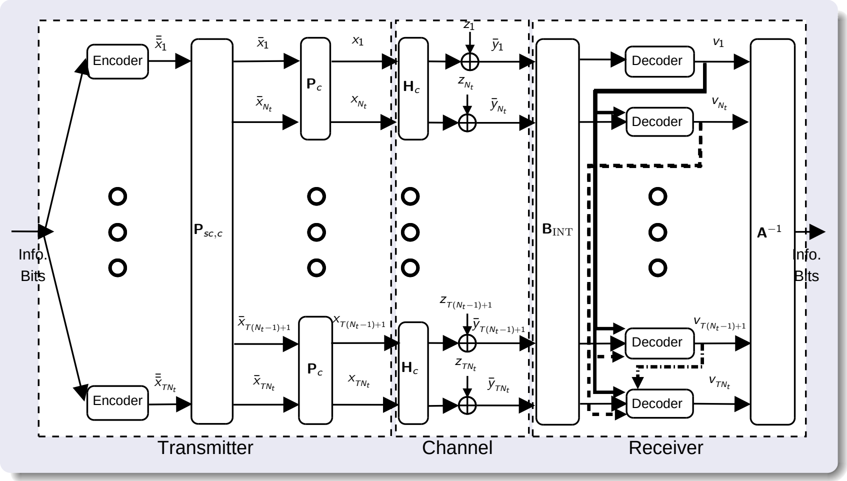
$$\bar{\mathbf{y}}_c^P = \mathcal{H}_c^P \mathbf{P}_{st,c} \bar{\mathbf{x}}_c + \bar{\mathbf{z}}_c$$

where

$$\mathcal{H}_c^P = \mathbf{I}_{T \times T} \otimes \mathbf{H}_c \mathbf{P}_c = \begin{bmatrix} \mathbf{H}_c\mathbf{P}_c & 0 & \cdots & 0 \\ 0 & \mathbf{H}_c\mathbf{P}_c & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \mathbf{H}_c\mathbf{P}_c \end{bmatrix}$$

- We assume that both precoding matrices are unitary

Space-Time Precoding



Space-Time Precoding

- WI-MI of this channel (normalized per channel use)

$$\frac{1}{T}\log\det\left(\mathbf{I}+(\mathcal{H}_c^P\mathbf{P}_{st,c})(\mathcal{H}_c^P\mathbf{P}_{st,c})^H\right)=C_{\mathrm{WI}}(\mathbf{H}).$$

- WC scheme outage is defined as

$$P_{\mathrm{out}}^{\mathrm{WC}}(C_{\mathrm{WI}},R)=\sup_{\mathbf{H}_c\in\mathbb{H}(C_{\mathrm{WI}})}P\left(\frac{1}{T}R_{\mathrm{IF}}(\mathcal{H}_c^P\cdot\mathbf{P}_{st,c})<R\right),$$

- ε-outage capacity $R(\mathbf{P}_{st,c};\varepsilon)$ is defined as the rate for which

$$P_{\mathrm{out}}^{\mathrm{WC}}\left(C_{\mathrm{WI}},R_{\mathrm{IF}}(\mathbf{P}_{st,c};\varepsilon)\right)=\varepsilon.$$

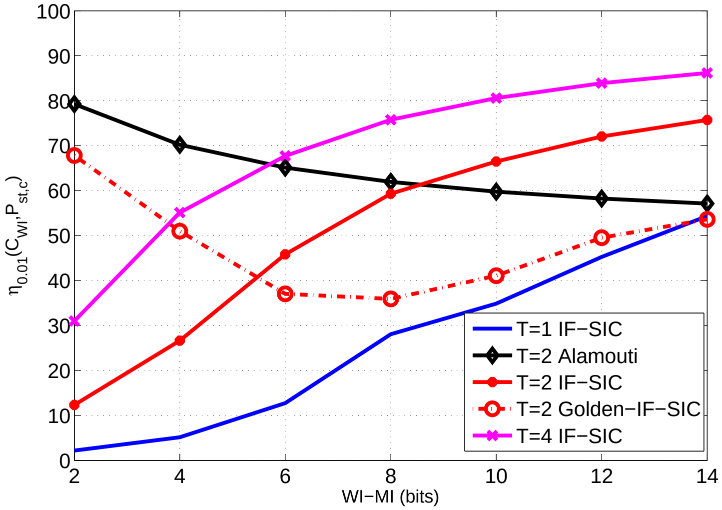
- The transmission efficiency is defined as

$$\eta_\varepsilon(C_{WI},\mathbf{P}_{st,c})=\frac{R_{\mathrm{IF}}(\mathbf{P}_{st,c};\varepsilon)}{C_{WI}}.$$

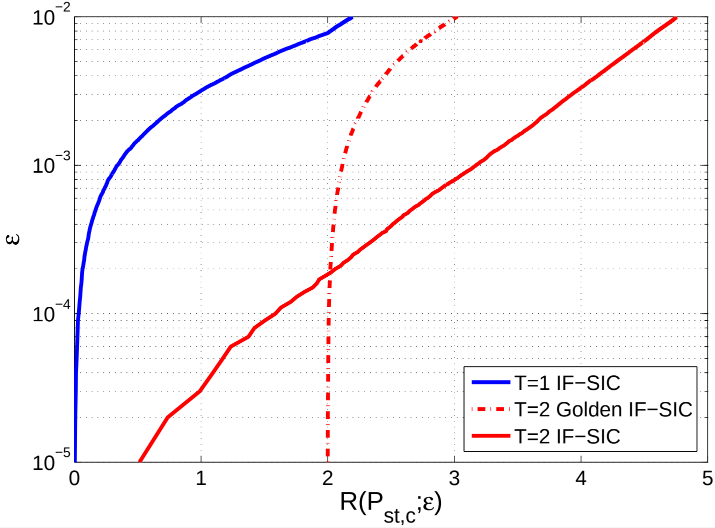
Space-Time Precoding

- Candidate precoding schemes
  - Orthogonal space-time block code (IF becomes superfluous)
  - Algebraic space-time block codes
    - 2×2 Golden
    - 4×4 Perfect code, punctured perfect code, MIDO
  - Random space-time block code
    - $\mathbf{P}_{st,c}$ is drawn from the CUE (hence $\mathbf{P}_c$ is redundant)

Space-Time Precoding: 2 Tx Antennas



A Closer Look at Random vs. Algebraic Space-Time Rotation



Upper Bound via ML

- ML decoder where each stream is coded using an independent Gaussian codebook
- Let $\mathbf{H}_S$ denote the submatrix of $\mathcal{H}_c^P \mathbf{P}_{st,c}$ formed by taking the columns with indices in $S \subseteq 1, 2, \ldots, N_t T$

$$R_{\text{JOINT,ST}} = \frac{1}{T} \min_{S \subseteq 1,2,\ldots,N_t T} \frac{N_t T}{|S|} \log\det\left(\mathbf{I}_S + \mathbf{H}_S \mathbf{H}_S^H\right)$$

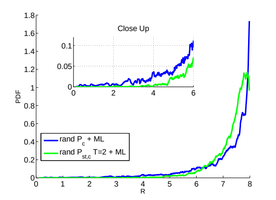
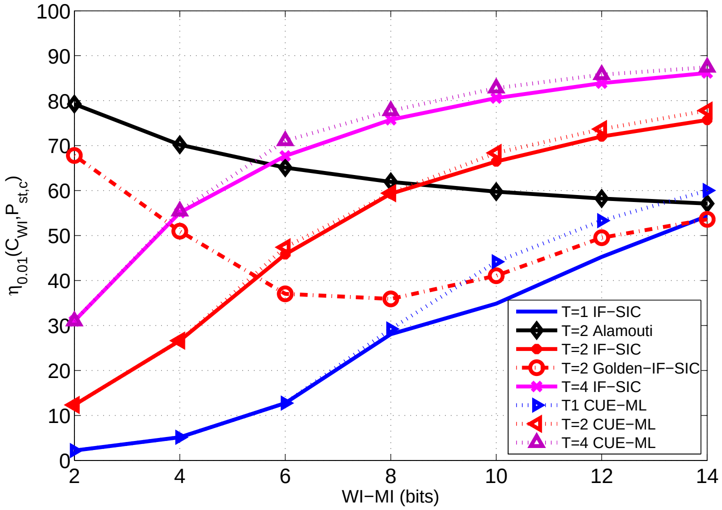


Figure: Approximate WC PDF (Monte carlo simulation) of joint ML

Space-Time Precoding: 2 Tx Antennas



A Lower Bound on the performance of ML for $Nr \times 2$

$$R_{\text{JOINT,ST}} = \frac{1}{T} \min_{S \subseteq 1,2,\ldots,N_t T} \frac{N_t T}{|S|} \log\det\left(\mathbf{I}_S + \mathbf{H}_S \mathbf{H}_S^H\right)$$

$$\mathcal{H}_c^P \mathbf{P}_{st,c} = \mathbf{U}\mathbf{D}\mathbf{V}^H$$

$$\begin{bmatrix} D1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & D1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D2 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & D2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} & \cdots & \mathbf{V}_{1,N_t T} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{V}_{\frac{N_t T}{2},1} & \mathbf{V}_{\frac{N_t T}{2},2} & \cdots & \mathbf{V}_{\frac{N_t T}{2},N_t T} \\ \mathbf{V}_{\frac{N_t T}{2}+1,1} & \mathbf{V}_{\frac{N_t T}{2}+1,2} & \cdots & \mathbf{V}_{\frac{N_t T}{2}+1,N_t T} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{V}_{N_t T,1} & \mathbf{V}_{N_t T,2} & \cdots & \mathbf{V}_{N_t T,N_t T} \end{bmatrix}$$

A Lower Bound on the performance of ML for $Nr \times 2$

$$R_{\mathrm{JOINT,ST}} = \frac{1}{T} \min_{S \subseteq 1,2,\ldots,N_t T} \frac{N_t T}{|S|} \log\det\left(\mathbf{I}_S + \mathbf{H}_S \mathbf{H}_S^H\right)$$

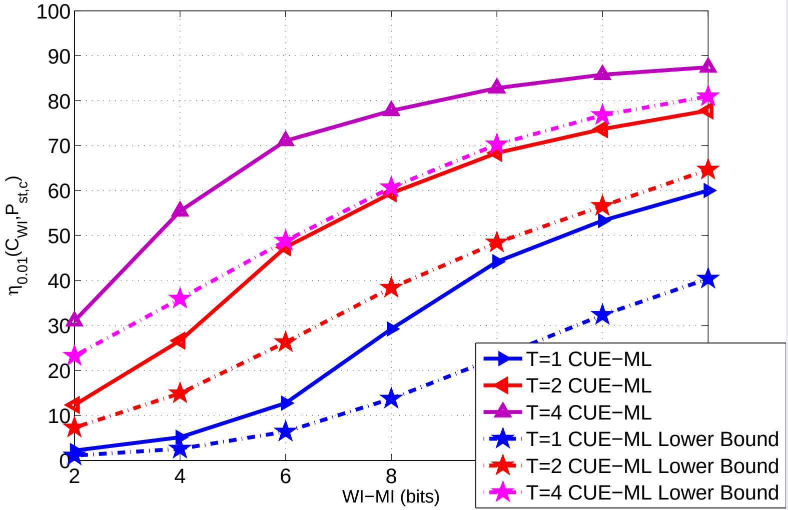
$$\mathcal{H}_c^P \mathbf{P}_{st,c} = \mathbf{U}\mathbf{D}\mathbf{V}^H$$

$$\begin{bmatrix} D1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & D1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & D2 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & D2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} & \cdots & \mathbf{V}_{1,N_t T} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{V}_{\frac{N_t T}{2},1} & \mathbf{V}_{\frac{N_t T}{2},2} & \cdots & \mathbf{V}_{\frac{N_t T}{2},N_t T} \\ \mathbf{V}_{\frac{N_t T}{2}+1,1} & \mathbf{V}_{\frac{N_t T}{2}+1,2} & \cdots & \mathbf{V}_{\frac{N_t T}{2}+1,N_t T} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{V}_{N_t T,1} & \mathbf{V}_{N_t T,2} & \cdots & \mathbf{V}_{N_t T,N_t T} \end{bmatrix}$$

A Lower Bound on the performance of ML for $Nr \times 2$

- When $D1 \neq D2$ columns are not orthogonal...
- Sketch of theorem (lower bound on ML performance)
  - Each sub-matrix is a part of a unitary matrix
  - The eigenvalues of this sub-matrix (taken from square unitary matrix) has a Jacobi distribution
  - We use a bound on the determinant of the sum of positive definite matrices
  - We use union bound to overcome dependence between two sub matrices
  - For a given S, we use union bound to cover all options to select S columns
  - We go over all options for S and take the minimum
- The outcome is a closed form expression that can be calculated numerically

A Lower Bound on the performance of ML for $Nr \times 2$



Space-Time Precoding: 4 Tx Antennas